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OPTIMUM FLIGHT PATHS OF TURBOJET AIRCRAFT

By Angelo Miele

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SUMMARY

The climb of turbojet aircraft is analyzed and discussed including the accelerations. Three particular flight performances are examined: minimum time of climb, climb with minimum fuel consumption, and steepest climb.

The theoretical results obtained from a previous study are put in a form that is suitable for application on the following simplifying assumptions: The Mach number is considered as independent variable instead of the velocity; the variations of the airplane mass due to fuel consumption are disregarded; the airplane polar is assumed to be parabolic; the path curvatures and the squares of the path angles are disregarded in the projection of the equation of motion on the normal to the path; lastly, an ideal turbojet with performance independent of the velocity is involved.

The optimum Mach number for each flight condition is obtained from the solution of a sixth order equation in which the coefficients are functions of two fundamental parameters: the ratio of minimum drag in level flight to the thrust and the Mach number which represents the flight at constant altitude and maximum lift-drag ratio.

Diagrams for a quick calculation of the optimum Mach numbers and the effect of accelerations on the rate of climb in tropospheric and stratospheric flight are given.

The critical discussion of the effect of the basic assumptions is followed by the suggestion of a method for taking into account the dependence of the turbojet performance from the velocity and the variations of the airplane mass due to fuel consumption.

The study is concluded with a numerical example which shows the unusual advantages accruing from the new flight technique when the climb is pushed to near ceiling level.

*"Traiettorie Ottime Di Volo Degli Aeroplani Azionati Da Turbo-reattori." L'Aerotecnica, n. 4, vol. XXXII, 1952, pp. 206-219.

SYMBOLS

$a = \sqrt{KR\tau}$	velocity of sound, m/s
C_p	lift coefficient
$C_{pE} = \sqrt{\pi\lambda e C_{r0}}$	lift coefficient representing maximum lift/drag ratio
C_r	drag coefficient
C_{r0}	drag coefficient at zero lift
e	Oswald efficiency factor
$E_{max} = \sqrt{\frac{\pi\lambda e}{4C_{r0}}}$	maximum aerodynamic efficiency
$F_a = V_z/V_{zu} = \sin \theta / \sin \theta_u$	acceleration factor = ratio of effective rates of climb to that computed with accelerations disregarded
g	acceleration of gravity, 9.8066 m/sec ²
K	ratio of specific heat at constant pressure to specific heat at constant volume, 1.4 for air
$M = V/a$	Mach number
$M_c = V_c/a$	Mach number at maximum rate of climb with accelerations disregarded
$M_r = V_r/a$	Mach number at maximum path angle computed with accelerations discounted
$M_E = V_E/a$	Mach number representing horizontal flight at maximum lift-drag ratio
N	rps of turbojet
p	pressure, kg/m ²
P	lift, kg

q	specific fuel consumption, kg/sec
Q	weight of airplane, kg
$R = p/\rho r$	constant of the air, $287.1 \text{ m}^2\text{sec}^{-2}\text{K}^{-1}$
R	aerodynamic resistance, kg
$R_{\min} = Q/E_{\max}$	minimum aerodynamic resistance in straight horizontal flight, kg
S	wing area, m^2
T	thrust, kg
V	speed, m/sec
V_c	speed at maximum V_{zu} , m/sec
V_r	speed at maximum θ_u , m/sec
V_E	speed of straight horizontal flight at maximum efficiency, m/sec
V_z	effective rate of climb, m/sec
V_{zu}	rate of climb with accelerations disregarded, m/sec
$x = R_{\min}/T = Q/TE_{\max}$	minimum drag/thrust ratio
Y	ratio of optimum speed for a certain flight condition with accelerations allowed for to the corresponding value determined with accelerations discounted
Z	altitude, m
Z_*	altitude of tropopause, 10,769 m
α	absolute value of temperature gradient in the troposphere, 0.0065°K/m
ϵ_c	error introduced in the calculation of the speed of climb due to disregarded centripetal acceleration
ϵ_t	error introduced in the calculation of the speed of climb as a result of disregarded tangential acceleration

ϵ_θ	error introduced in the calculation of the speed of climb as a result of having assumed $\cos \theta = 1$ in the projection of the equation of motion on the normal to the flight path
θ	angle of effective path
θ_u	path angle computed while disregarding the accelerations
λ	geometrical aspect ratio
ρ	air density, $\text{kg sec}^2 \text{m}^{-4}$
$\sigma = \rho/\rho_0$	relative air density
τ	absolute temperature, $^\circ\text{K}$
Subscripts:	
0	refers to sea level
*	refers to tropopause

1. INTRODUCTION

A general method relating to problems of optimum in nonsteady flight was developed and discussed in earlier reports (refs. 15 and 19). They define the particular distributions of the velocity $V = V(Z)$ which produce the initial conditions (V_1, Z_1) for prescribed final conditions (V_2, Z_2) with minimum time or minimum fuel consumption or minimum distance covered in horizontal flight.

The weight of the airplane is assumed constant. The angle between thrust and velocity vectors is disregarded. The projection of the equation of motion on the normal to the flight path is approximated at $P = Q$. For the steepest climb, the additional assumptions $\sin \theta \cong \tan \theta$ in the projection of the equation of motion on the tangent to the flight path are accepted.

For the thrust and the fuel consumption per unit time, the analytic expressions

$$T = T(V, Z) \quad (1)$$

$$q = q(V, Z) \quad (2)$$

are assumed.

It is shown that the optimum distributions $V = V(Z)$ include two terminal paths depending on the limiting conditions and a central section defined by

(a) Climb with minimum time

$$\frac{\partial(TV - RV)}{\partial V} = \frac{V}{g} \frac{\partial(TV - RV)}{\partial Z} \quad (3)$$

(b) Climb with minimum fuel consumption

$$\frac{\partial}{\partial V} \left[\frac{TV - RV}{q} \right] = \frac{V}{g} \frac{\partial}{\partial Z} \left[\frac{TV - RV}{q} \right] \quad (4)$$

(c) Steepest climb

$$\frac{\partial(T - R)}{\partial V} = \frac{V}{g} \frac{\partial(T - R)}{\partial Z} \quad (5)$$

In the most important flight condition (with the same nomenclature as used in ref. 19), the first case (acceleration and climb at low speed and low altitude, at high speed and high altitude) and under the restrictive conditions $Z_1 \leq Z \leq Z_2$, the foregoing terminal paths result in two horizontal accelerated motions corresponding to the extreme altitudes Z_1, Z_2 .

To illustrate: For the rapid climb (equation (3) is synthetically indicated by $\omega(V, Z) = 0$), the following theoretical optima in flight are indicated:

(a) Terminal altitudes, both tropospheric or both stratospheric (fig. 1-A):

(1) Accelerating at constant altitude Z_1 from V_1 to V_M defined with $\omega(V, Z_1) = 0$.

(2) Climb from Z_1 to Z_2 utilizing the velocity distribution defined by $\omega(V, Z) = 0$.

(3) Accelerating at constant altitude Z_2 from V_N defined by $\omega(V, Z_2) = 0$ to final speed V_2 .

(b) Initial tropospheric altitude and final stratospheric altitude (fig. 1-B):

(1) Accelerations at constant altitude Z_1 from V_1 to V_M defined by $\omega(V, Z_1) = 0$.

(2) Climb from (Z_1, V_M) to (Z_*, V_t) using the velocity distribution defined by $\omega(V, Z) = 0$.

(3) Accelerations corresponding to the tropopause from V_t to V_S .

(4) Climb from (Z_*, V_S) to (Z_2, V_N) using the velocity distribution $\omega(V, Z) = 0$.

(5) Accelerations at constant altitude Z_2 from V_N to final speed V_2 .

2. TURBOJET AIRCRAFT

The values developed in the foregoing are of general character. In the present paper, it is intended to apply them to the specific case of turbojet aircraft by transforming the data already available with the aid of suitable assumptions for more convenient use.

Thrust and fuel consumption per unit time of turbojets are, in standard atmosphere, functions of the following nature:

$$T = T(V, Z, N) \quad (6)$$

$$q = q(V, Z, N) \quad (7)$$

V the velocity, Z the altitude, and N the number of revolutions per minute.

Accordingly, the velocity distributions (3), (4), and (5) produce optimum values for the respective cases of flight, in correspondence with each arbitrarily prescribed function

$$N = N(V, Z) \quad (8)$$

which makes it possible to reduce (6) and (7) to (1) and (2).

In accordance with ordinary operational practice, the paths flown at constant number of revolutions are analyzed. Accordingly (8) is replaced by

$$N = \text{constant} \quad (9)$$

3. ADDITIONAL ASSUMPTIONS RELATING TO THE POWER PLANT

In order to obtain simple solutions, an ideal turbojet with performance independent of the velocity¹ is assumed. Accordingly, having assumed $N = \text{constant}$, thrust and fuel consumption depend solely on the altitude. The following analytical representation is preferred:

(a) Troposphere

$$T/T_0 = \sigma^r = f_1(Z) \quad (10)$$

$$q/q_0 = \sigma^s = f_2(Z) \quad (11)$$

(b) Stratosphere

$$T/T_* = q/q_* = \sigma/\sigma_* = f_3(Z) \quad (12)$$

A statistical analysis of the performances of a number of turbojets has proved the suitability of taking the following medium values

$$r = 0.75 \quad s = 0.90 \quad (13)$$

into consideration for the conditions of climb.

They are only approximate and differ somewhat from those defined by Ashkenas (ref. 7) and Tifford (ref. 8) in previous studies where the principal object was the study of the range performance of turbojet airplanes.

4. ADDITIONAL ASSUMPTIONS RELATING TO THE AIRCRAFT

The aircraft drag polar is assumed to be parabolic

$$C_r = C_{r0} + C_p^2/\pi\lambda e \quad (14)$$

with coefficients C_{r0} , e independent of the Mach number and the Reynolds number.

¹A method for taking the dependence of T and q on the speed is given at the end of the present report.

Having approximated the projection of the equation of motion on the normal to the flight path with

$$Q = P = \frac{1}{2} C_p \rho S V^2 \quad (15)$$

it follows that the aerodynamic resistance takes the form

$$R = \frac{1}{2} C_{r0} \rho S V^2 = E \sigma V^2 + F / \sigma V^2 \quad (16)$$

with

$$E = \frac{1}{2} C_{r0} \rho_0 S \quad (17)$$

$$F = 2Q^2 / \pi \lambda e \rho_0 S \quad (18)$$

5. OPTIMUM VELOCITY

Development of equations (3), (4), and (5) with the aid of (10), (11), (12), and (16) results in the following sixth power equation

$$AV^6 + BV^4 + CV^2 + 1 = 0 \quad (19)$$

The coefficients A, B, and C depend on the type of flight and altitude, as well as on the characteristic parameters of the aircraft and the engine. Hence:

(a) Climb - minimum time

$$A = \frac{E\sigma}{g^F} \frac{d\sigma}{dZ} \quad (20)$$

$$B = - \frac{3E\sigma^2}{F} - \frac{\sigma}{g^F} \frac{dT}{dZ} \quad (21)$$

$$C = \frac{T\sigma}{F} - \frac{1}{g\sigma} \frac{d\sigma}{dZ} \quad (22)$$

(b) Climb - minimum fuel consumption

$$A = \frac{E\sigma^2}{g^F} \left[\frac{1}{\sigma} \frac{d\sigma}{dZ} - \frac{1}{q} \frac{dq}{dZ} \right] \quad (23)$$

$$B = -\frac{3E\sigma^2}{F} + \frac{T\sigma}{gF} \left[\frac{1}{q} \frac{dq}{dZ} - \frac{1}{T} \frac{dT}{dZ} \right] \quad (24)$$

$$C = \frac{T\sigma}{F} - \frac{1}{g} \left[\frac{1}{\sigma} \frac{d\sigma}{dZ} + \frac{1}{q} \frac{dq}{dZ} \right] \quad (25)$$

(c) Steepest climb

$$A = \frac{E\sigma}{2gF} \frac{d\sigma}{dZ} \quad (26)$$

$$B = -\frac{E\sigma^2}{F} - \frac{\sigma}{2gF} \frac{dT}{dZ} \quad (27)$$

$$C = -\frac{1}{2g\sigma} \frac{d\sigma}{dZ} \quad (28)$$

6. FUNDAMENTAL PARAMETERS

The following variables are introduced:

(a) Ratio of minimum drag in straight horizontal flight to thrust

$$x = R_{\min}/T = Q/T_{E_{\max}} \quad (29)$$

(b) Speed in straight horizontal flight in attitude of maximum efficiency

$$V_E = \sqrt[4]{\frac{F}{E\sigma^2}} = \sqrt{\frac{2Q}{C_{pE}\rho S}} \quad (30)$$

and the corresponding Mach number

$$M_E = V_E/a \quad (31)$$

These parameters or other derivatives of their combinations are fundamental in the study of all flight performances of jet aircraft (refs. (4), (5), (13), and (14)). They allow for the basic aerodynamic characteristics (λ_e , C_{r0}), wing loading, altitude of flight, and excess of thrust with respect to the minimum drag.

7. OPTIMUM MACH NUMBER

The use of the results obtained in section 5 presents no conceptional difficulties, but the calculations involve much paper work. Fortunately, however, it is possible to effectuate considerable simplifications by choosing the Mach number instead of the velocity as basic variable and introducing the fundamental parameters M_E and x .

Equations (10), (11), and (12), and that of the standard atmosphere give:

(a) Troposphere

$$\frac{1}{\sigma} \frac{d\sigma}{dz} = \frac{1}{rT} \frac{dT}{dz} = \frac{1}{sq} \frac{dq}{dz} = - \frac{\alpha \epsilon}{r} \quad (32)$$

with

$$\epsilon = \frac{g}{\alpha R} - 1 \quad (33)$$

(b) Stratosphere²

$$\frac{1}{\sigma} \frac{d\sigma}{dz} = \frac{1}{T} \frac{dT}{dz} = \frac{1}{q} \frac{dq}{dz} = - \frac{g}{Rr} \quad (34)$$

Furthermore, by defining

$$m = 1 - \frac{\alpha R}{g} \quad (35)$$

and bearing in mind the expression of the velocity of sound and equations (17), (18), (20), (21) . . . (35), equation (19) can be transformed into

$$A_1 M^6 + B_1 M^4 + C_1 M^2 + 1 = 0 \quad (36)$$

with

$$A_1 = A_2^6 = K_1 / M_E^4 \quad (37)$$

²Obviously equation (34) can be obtained from (32) and (33) by putting $r = s = 1$ and $\alpha = 0$.

$$B_1 = Ba^4 = K_2/M_E^4 + K_3/xM_E^2 \quad (38)$$

$$C_1 = Ca^2 = K_4/xM_E^2 + K_5 \quad (39)$$

The factors $K_1 \dots K_5$ are represented in table 1.

Equation (36) as represented in figures 2, 3, 4, 5, 6, and 7, enables a quick determination of the optimum Mach number (M) for a given type of flight when the basic parameters M_E and x are known. The same graphs also show the ratio (Y) of the Mach number that solves the question of optimum allowing for accelerations, and that Mach number which will solve the problem when the accelerations are disregarded. The latter Mach number gives

$$M_c = \sqrt{\frac{1 + \sqrt{1 + 3x^2}}{3x}} M_E \quad (40)$$

for the climb with minimum time and minimum fuel consumption, and

$$M_r = M_E \quad (41)$$

for the steepest climb.³

To simplify the calculation of the optimum velocity distribution, the functions $M_E/M_{E0} = f(Z)$, $x/x_0 = f(Z)$ are reproduced in figure 8 and table 2. They are defined

(a) Troposphere

$$\frac{M_E}{M_{E0}} = \sigma^{-0.617} \quad (42)$$

$$\frac{x}{x_0} = \sigma^{-0.75} \quad (43)$$

(b) Stratosphere

$$\frac{M_E}{M_{E0}} = 1.149\sigma^{-0.5} \quad (44)$$

³It is interesting to note that the equations (40) and (41) can be obtained as particular cases of (36), if the terms deriving from the presence of the accelerations are made zero, thus by putting $K_1 = K_3 = K_5 = 0$.

$$\frac{x}{x_0} = 0.744\sigma^{-1} \quad (45)$$

It is appropriate to point out that, as far as the turbojet is concerned, optimum velocity or Mach number at a given altitude is influenced primarily by the local value of the thrust and to a lesser extent by the derivative of T and q with respect to the altitude. Hence, it follows that whenever the exponents of the laws of variations of T and q depart from the mean value (13), the results obtained can be considered applicable in good approximation. Greater accuracy is obtained by determining the parameter x at different altitudes on the basis of (29) instead of (43) and (45), and the effective values of the thrust which can be read from the operating charts of the turbojet. Therefore it is advisable to use (43) and (45) (and fig. 8 and table 2) only when the altitude performance is not known or when it is necessary to make a quick prediction of a preliminary project.

8. SPEED OF CLIMB

The climbing speed is (ref. 19)

$$V_z = F_a V \sin \theta_u = F_a M a \sin \theta_u \quad (46)$$

In equation (46), $\sin \theta_u$ indicates the path angle with accelerations neglected. Based on equations (16), (17), (18), (29), (30), and (31)

$$\sin \theta_u = \frac{T - R}{Q} = \frac{T}{Q} \left\{ 1 - \frac{x}{2} \left[\left(\frac{M}{M_E} \right)^2 + \left(\frac{M_E}{M} \right)^2 \right] \right\} \quad (47)$$

The factor of acceleration F_a is the ratio of effective rate of climb to that computed with accelerations discounted

$$F_a = \frac{V_z}{V_{zu}} = \frac{\sin \theta}{\sin \theta_u} = \frac{1}{1 + \frac{1}{2g} \frac{dV^2}{dZ}} \quad (48)$$

From equation (48), together with the expressions giving the optimum velocity distributions, follows

$$F_a = \frac{1}{1 + M^2 \frac{\phi(z, M_E)}{\psi(x, M_E)}} = F(M, M_E) \quad (49)$$

with

$$\Phi(x, M_E) = (C_1 M^4 + C_2 M^2)x + (C_3 M^2 + C_4)M_E^2 + C_5 x M_E^4 \quad (50)$$

$$\Psi(x, M_E) = (C_6 M^4 + C_7 M^2)x + (C_8 M^2 + C_9)M_E^2 + C_{10} x M_E^4 \quad (51)$$

The coefficients $C_1 \dots C_{10}$ for tropospheric flight are given in table 3. The values relating to stratospheric flight are obtained from those preceding by putting

$$\begin{aligned} m &= r = s = 1 \\ \epsilon &= \infty \end{aligned} \quad (52)$$

Equation (49) is plotted in figures 9, 10, 11, and 12 for optimum climb with minimum time and minimum fuel consumption.

It should be noted that equation (49) contains as particular case, the formulas of the acceleration factors associated with the distributions (40) and (41). It is sufficient to put $C_1 = C_3 = C_5 = C_6 = C_8 = C_{10} = 0$.

9. NUMERICAL APPLICATIONS

As a check on the actual advantages obtainable from the use of the deduced velocity distributions, the analysis is made on a numerical example (case I).

Time, space, and fuel consumption connected with the optimum velocity distributions are compared with the time, space, and consumption associated with a certain number of arbitrary distributions.

Eight velocity distributions $V = V(Z)$ are considered (fig. 13). Each one includes two accelerated motions corresponding to the initial and final altitude and a center section of the flight path along which the function $V = V(Z)$ is defined as follows:

(A) Distribution - minimum time (eq. (3)).

(B) Distribution - minimum fuel consumption (eq. (4)).

(C) Distribution - minimum distance flown horizontally (eq. (5)).

(D) Velocity distribution for maximum rate of climb at each altitude, accelerations disregarded (eq. (40)).

(E) Velocity distribution for maximum path angle at every altitude, accelerations disregarded (eq. (41)).

(F) Arbitrary distribution defined by $V_{kmh} = 396 + 21.0 Z_{km}$.

(G) Arbitrary distribution defined by $V_{kmh} = 650 + 11.4 Z_{km}$.

(H) Arbitrary distribution defined by $V_{kmh} = \text{const.} = 805$.

The characteristics of the airplane in question are:

$$Q = 5000 \text{ kg} \quad C_{r0} = 0.018$$

$$S = 25 \text{ m}^2 \quad T_0 = 1680 \text{ kg}$$

$$\lambda = 5 \quad q_0 = 0.606 \text{ kg/sec}$$

$$e = 0.8$$

The theoretical ceiling is about 14,600 m. Thrust and specific fuel consumption are supposed to vary with the altitude in accord with equations (10), (11), and (12). The dependence of T and q on velocity is disregarded in the present example.

The limiting conditions are

(1) Initial: $Z = 0 \text{ km}$, $V = 274 \text{ kmh}$.

(2) Final: 9.15 km, 10.769 km, 12.2 km, 13.7 km altitude, but the same final speed ($V = 805 \text{ km/h}$).

The principal numerical results are condensed in tables 4, 5, 6, 7, 8, and 9. The basic conclusions are as follows:

(a) Rate of climb - minimum time. - The relative and absolute importance of using the new unsteady distribution A instead of the old steady distribution D increases with the final altitude. The gain in time is merely 3.4 seconds (0.57 percent) for a climb to 9.15 km, but 116.5 seconds (7.64 percent) for a climb to 13.7 km. These figures are in agreement with those obtained by Lush (ref. 17).

It should be noted that the distribution A is better than D in the overall sense but not in the partial. Thus table 8 shows, for example, that an airplane in flight using distribution A recovers during the accelerated horizontal phase of motion the time lost at final altitude with respect to distribution D during the other phases of flight.

(b) Rate of climb - minimum fuel consumption.- The qualitative conclusions are identical with those for minimum time, but the advantages accruing from using unsteady distribution B instead of D are less (the saving in fuel for a 13.7 km climb is 2.5 percent).

From the practical point of view, it is interesting to note the possibility of obtaining a good compromise between the necessity of a climb in minimum time and climb with minimum fuel consumption.

(c) Steepest climb.- A comparison of the unsteady solutions (C) with the steady (E) for this kind of flight yields conclusions similar to those obtainable for the climb in minimum time.

10. CRITICAL REMARKS REGARDING THE OBTAINED SOLUTIONS

(a) Variation of thrust and fuel consumption with the altitude.- The mean values of $r(0.75)$ and $s(0.90)$ involved are merely approximate. Nevertheless, the writer believes that the practical divergence of r and s from the assumed values have a negligible effect on his results, if the advice given at the end of section 7 is taken.

(b) Compressibility effect.- The foregoing formulas were deduced while disregarding the dependence of the aerodynamic parameters of the aircraft (C_{r0} , e) from the Mach number. In consequence, they apply when the Mach number of flight is lower than the critical Mach number corresponding to the lift coefficient known from the condition of flight in question. From the qualitative point of view, it is permissible to predict that the compressibility tends to produce an approach between the unsteady and the corresponding steady distributions. However, they reduce the advantages obtainable by the use of the new flight techniques.

(c) Discontinuities in the solution at the tropopause.- The equations defining the optimum velocity have two solutions representing the tropopause (ref. 19). One is the terminal velocity of the tropospheric flight, the other is the initial velocity of the stratospheric flight. It can be proved analytically that stratospheric flight is always greater than the tropospheric at $Z = Z_*$.

(d) Comparison of steady and unsteady solutions.- The figures 2, 3, 4, 5, 6, 7, and 13 prove that the optimum velocity obtained by the present analysis is, in general, greater than those obtained by a "steady" analysis. In the case of the climb with minimum fuel consumption, the results may be not as good (low altitude, $x < 0.379$).

(e) Effect of the accelerations on the speed of climb.- The figures 9, 10, 11, 12, and table 9 (example treated in section 9) show the need for

including the accelerations in the evaluation of the climb performance. The significance of this concept increases with increasing altitude, especially in the stratosphere.

(f) Dependence of the turbojet performance on the altitude.- The aspect of an ideal turbojet with performance independent of V was basic to the effects of obtaining easily usable results. Whenever it is desired to include the changes in T and q , it is advisable to proceed in the following approximate manner. Reference is made to the climb with minimum fuel consumption which is the type of flight for which this note can be of particular interest. The other cases are treated by analogy.

The optimum velocity is obtained:

(1) by computing the steady solution, that is, the velocity which solves

$$\frac{\partial}{\partial V} \left[\frac{TV - RV}{q} \right] = 0 \quad (53)$$

The dependence of T and q on V is considered.

(2) by multiplying the preceding velocity by the factor

$$Y = V/V_c = M/M_c = f(x, M_E) \quad (54)$$

which allows for the accelerations and can be deduced from figures 4 and 5.

(g) Effect of change of airplane mass on climb performance.- The foregoing results were obtained by considering an airplane of constant ideal mass; however, due to the fuel consumption, a correction can be applied at some suitable time. The accelerations and mass changes of the aircraft ordinarily exert opposite but noncompensating effects on the flight performance. At low altitude, the effect is unfavorable due to the accelerations. At high altitude, the effect is the opposite. In particular, the theoretical ceiling is higher than computed by assuming $Q = \text{const}$. It is somewhat difficult to study the conditions of the optimum when the changes of Q are involved; however, fortunately a satisfactory solution from the point of view of modern engineering practice can be obtained by proceeding iteratively (ref. 19) as follows:

(a) Compute in first approximation the fuel consumption for $Q = \text{const}$.

(b) Determine the instantaneous weight of the airplane at various altitudes.

(c) Introduce the optimum velocities and the climb velocity corresponding to the instantaneous values of Q in the equations.

The convergence is very rapid. Two attempts are sufficient. The principal conclusions are as follows: The mass changes of the aircraft due to the consumed fuel produce an increase of V_z , and a decrease in the optimum velocity relative to the computed values on the basis of $Q = \text{const.}$, the first effect (on V_z) having the greatest relative significance. It is readily apparent on the calculation of the airplane of section 9 and the climb in minimum time. (Compare table 10.)

11. CONCLUSIONS

The principal results can be summed up as follows:

- (1) The problem of the turbojet aircraft relative to the velocity distribution $V = V(Z)$ is solved. It results in certain initial conditions and prescribed final conditions with minimum time, minimum fuel consumption, and minimum flown horizontal distance.
- (2) When the mass changes of the airplane due to the consumption of fuel, the curvature, and squares of the path angle are disregarded, the optimum technique of flight yields two terminal flight paths depending on the limiting conditions and a central path along which the velocity distribution is defined by equations (3), (4), and (5).
- (3) In flight case I (acceleration and climb at low speed and low altitude at high speed and high altitude) and under the restrictive conditions imposed on the flight path to remain within the area of the space limited by the horizontal planes corresponding to the extreme altitudes, the two foregoing terminal paths are two paths to be flown in level horizontal flight.
- (4) The optimum velocity distributions along the central path flown at variable altitude are defined by equations of the sixth power, provided the polar is assumed parabolic and the operation is carried out at constant number of revolutions with an ideal turbojet of performance independent of the velocity.
- (5) It is advisable to assume the Mach number of velocity as fundamental variable, since the solutions obtained will then be dependent solely on two parameters: the Mach number corresponding to rectilinear horizontal flight in position of maximum efficiency and the minimum drag to thrust ratio. Graphs are supplied for each case which permit a quick calculation of the optimum Mach numbers and of the effect of the accelerations on the speed of climb.
- (6) Climb - minimum time. - The optimum velocities are always greater than those obtained from a "steady" analysis. The relative difference

may reach up to 20 percent in the stratosphere. The accelerations can reduce the climbing speed up to 20 percent in stratospheric flight.

(7) Climb with minimum fuel consumption.- The optimum velocities for stratospheric flight are always greater than those obtainable with the accelerations disregarded. In tropospheric flight, they can be less at low altitudes ($x < 0.379$; $T > 2.64 R_{min}$). The accelerations can reduce the rates of climb by as much as 30 percent in the stratosphere.

(8) Steepest climb.- The optimum velocities are always greater than those obtained by "steady" analysis. The difference may amount to as much as 30 percent in the stratosphere. The effect of the accelerations on the speed of climb is more severe than in the preceding cases.

(9) The validity of the foregoing results is subject to a check that the Mach number of flight is lower than the critical Mach number corresponding to the lift coefficient relating to the conditions in question. The compressibility has an unfavorable effect on the climb performance. More particularly, an approach between the new unsteady distributions and the old steady distributions produces a decrease of the advantages obtainable through the use of the new techniques of flight.

(10) In the troposphere, the solutions are discontinuous. Corresponding to such altitude, it is necessary to accelerate the aircraft from its terminal velocity of tropospheric flight to the initial velocity of stratospheric flight, if the optimum climbing performances are to be obtained.

(11) The changes in airplane weight due to fuel consumption are taken care of by a correction applied to the second approximation. In consequence, the climbing speed is increased and the velocity on the path reduced with respect to the values computed for $Q = \text{const}$.

(12) The following concepts are determined by way of an example on a typical jet fighter:

(a) The relative and absolute significance of using the optimum flight path deduced from that obtained by disregarding the increased accelerations with increasing final altitude for every case of flight.

(b) As far as the climb in minimum time is concerned, the new unsteady solution, compared to the corresponding steady solution, makes it possible to save 7.6 percent of the total time (about 2 minutes) in a final climb to 13.7 km.

(c) A good compromise can be reached between the necessity of climb in minimum time and climb with minimum fuel consumption.

APPENDIX

JUSTIFICATION OF THE ASSUMPTIONS USED IN THE DEDUCTION
OF THE OPTIMUM VELOCITY DISTRIBUTIONS

(1) Climb - Minimum Time

In ordinary mechanics of flight, the optimum velocity for the climb is deduced like that for maximal velocity with accelerations disregarded. Additional assumptions are usually adapted which consist in disregarding the squares of the path angles in the projection of the equation of motion on the normal to the flight path; however, it is added to the steady solution (40).

The climbing speed computed on the basis of these assumptions and of the velocity on the flight path defined by (40) is affected by errors which are reflected in the evaluation of the time of climb and the consumption of fuel. These errors included in the calculation of V_z are itemized so as to show their absolute and relative significance. We shall omit details of the lengthy calculations based on the steady distribution (40) and on the assumptions used in the course of the work. Only the results are given here. The errors are given in relative form and referred to velocity V_{zu} .

(a) Error due to disregard of tangential accelerations. - It is expressed by

$$\epsilon_t = \frac{V_z - V_{zu}}{V_{zu}} = F_a - 1 = - \frac{1}{1 + \chi} \quad (55)$$

with

$$\chi = \frac{6x\xi}{Km(\xi - r)(1 + \xi)M_E^2} \quad (\text{troposphere}) \quad (56)$$

$$\chi = \frac{2\xi}{KxM_E^2} \quad (\text{stratosphere}) \quad (57)$$

$$\xi = \sqrt{1 + 3x^2} \quad (58)$$

This error increases in ratio to the altitude and reaches its highest value in correspondence with the theoretical ceiling ($x = 1$, $\xi = 2$)

$$\epsilon_t = - \frac{1}{1 + 4/KM_E^2} \quad (59)$$

From (59), it follows that the omission of the tangential accelerations can result in errors up to 20 percent of the climbing speed at altitudes near the ceiling.

(b) Error due to disregard of centripetal accelerations.— Troposphere

$$\epsilon_c \approx - \frac{2F}{g\sigma Q} \frac{d\theta_u}{dZ} \approx \frac{2Kmr(2\xi - 1)M_E^2}{3x\xi E_{\max}^2} \quad (60)$$

stratosphere

$$\epsilon_c = \frac{2K(2\xi - 1)M_E^2}{3x\xi E_{\max}^2} \quad (61)$$

This error increases with the altitude and reaches its highest value in correspondence to the theoretical ceiling height

$$\epsilon_c = \frac{KM_E^2}{E_{\max}^2} \quad (62)$$

It is easily checked that ϵ_c is usually below 0.005.

(c) Error due to disregard of squares of the path angles:

$$\epsilon_\theta \approx \frac{FV_{zu}}{Q\sigma V_c^3} = \frac{1}{E_{\max}^2} \frac{2 - \xi}{1 - \xi} \quad (63)$$

This error, unlike the other two, assumes greater significance at low altitude where the thrust is high. In the ideal limiting case, $x \rightarrow 0$, $\xi \rightarrow 1$, it results in

$$\epsilon_\theta = \frac{1}{2E_{\max}^2} \quad (64)$$

This formula produces much smaller errors than those given by Lush in reference (17) which are too conservative. Ordinarily, $\epsilon_\theta < 0.004$.

(d) Conclusions.— The application of the preceding formulas to modern jet aircraft shows that the error due to omission of the tangential accelerations is from 10 to 60 times greater than the total error due to disregarded centripetal accelerations and the squares of the path angle.

These statements justify the following conclusions: Of the classical assumptions made on the basis of the "steady" analysis of optimum conditions of jet airplane climb, the only one that gives rise to any appreciable effective error is that concerning the tangential accelerations. Hence, if the conditions of the optimum are analyzed while continuing to disregard the curvatures and the squares of the path angles but including the tangential accelerations, it must necessarily lead to more satisfactory results from the point of view of applications in engineering. In consequence, the assumptions leading to equation (3) remain justifiable.

(2) Climb - Minimum Fuel Consumption

Considering that the optimum distribution for the rate of climb with minimum consumption does not differ much from that of the minimum time, the considerations discussed in point (1) of the appendix are quantitatively applicable to this case of flight and the use of equation (4) is justified.

(3) Steepest Climb

The velocity distribution of the path angle computed while disregarding the accelerations and posing $\cos \theta = 1$ in the projection of the equation of motion on the normal to the flight path is defined by (41).

The errors introduced in the calculation of the climbing speed as a result of the classical assumptions are the following. They are expressed in relative form and referred to V_{zu} , this time computed on the basis of the distribution (41).

(a) Error due to disregard of tangential accelerations.- Troposphere

$$\epsilon_t = - \frac{1}{1 + 2/Km_E^2} \quad (65)$$

stratosphere

$$\epsilon_t = - \frac{1}{1 + 2/KM_E^2} \quad (66)$$

(b) Error due to disregard of centripetal accelerations.- Troposphere

$$\epsilon_c = \frac{K m r M_E^2}{x_E^2_{\max}} \quad (67)$$

stratosphere

$$\epsilon_c = \frac{K M_E^2}{x_E^2_{\max}} \quad (68)$$

(c) Error due to assuming $\cos \theta = 1$:

$$\epsilon_\theta = \frac{1}{2E^2_{\max}} \left[\frac{1}{x} - 1 \right] \quad (69)$$

A number of applications to modern turbojet aircraft has proved that the proportion of the error due to tangential accelerations is from 2 to 100 times greater than the combined error due to exclusion of centripetal accelerations and the squares of path angles.

The mode of calculation on which equation (5) is based remains applicable to high altitude as well as to low altitude, if the ratio thrust weight is not excessive. This is confirmed by the foregoing results and also by the study of the errors as a result of having assumed $\sin \theta \cong \tan \theta$ in the projection of the equation of motion on the tangent to the flight path.

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National Advisory Committee
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TABLE I.- COEFFICIENTS OF THE EQUATIONS THAT DEFINE
THE OPTIMUM MACH NUMBERS

Coefficients	Climb - minimum time		Climb - minimum fuel consumption		Steepest climb	
	Troposphere	Stratosphere	Troposphere	Stratosphere	Troposphere	Stratosphere
K_1	$-Km$	$-K$	$K(s - 1)m$	0	$-Km/2$	$-K/2$
K_2	-3	-3	-3	-3	-1	-1
K_3	$2Krm$	$2K$	$2Km(r - s)$	0	Krm	K
K_4	2	2	2	2	0	0
K_5	Km	K	$Km(s + 1)$	$2K$	$Km/2$	$K/2$

TABLE 2.- VALUES OF THE FUNCTIONS M_E/M_{E0} AND x/x_0

z_{km}	M_E/M_{E0}	x/x_0
0	1.000	1.000
2	1.129	1.159
4	1.282	1.352
6	1.465	1.591
8	1.687	1.888
10.769	2.077	2.432
12	2.288	2.950
14	2.676	4.035
16	3.130	5.520
18	3.661	7.551
20	4.281	10.331

TABLE 3.- COEFFICIENTS FOR COMPUTING THE ACCELERATION FACTOR F_a - TROPOSPHERE

Coefficient	Climb - minimum time	Climb - minimum fuel consumption	Steepest Climb
$C_1/K^2m^2\left(1 - \frac{1}{2\epsilon}\right)$	-1	$s - 1$	-1/2
$C_2/3Km$	-1	-1	-1/3
$C_3/K^2rm^2\left(1 + r - \frac{1}{\epsilon}\right)$	1	$1 - \frac{s}{r}$	1/2
$C_4/Km(1 + r)$	1	1	0
$2\epsilon C_5/K^2m^2$	-1	$-(1 + s)$	-1/2
$C_6/3Km$	-1	$s - 1$	-1/2
C_7	-6	-6	-2
$C_8/4Kmr$	1	$1 - \frac{s}{r}$	1/2
C_9	2	2	0
C_{10}/Km	1	$1 + s$	1/2

TABLE 4.- OPTIMUM VELOCITY DISTRIBUTIONS FOR CLIMBING FLIGHT

Z_{km}	M_E	$x=Q/\overline{TE}_{max}$	Optimum Mach numbers		
			Rate of Climb		
			Minimum time	Minimum consumption	Steepest
0	0.240	0.225	0.439	0.417	0.255
3.05	.290	.282	.480	.454	.311
6.10	.355	.361	.531	.501	.384
9.15	.441	.470	.600	.569	.484
10.769	.500	.547	.650	.615	.553
10.769	.500	.547	.688	.643	.599
12.2	.559	.684	.715	.683	.661
13.7	.630	.869	.750	.740	.736

TABLE 5.- TIME NECESSARY TO ACCELERATE AND CLIMB (sec)

Velocity distribution	Final altitude, km			
	9.15	10.769	12.2	13.7
A	597.5	755.0	976.9	1521.3
B	601.0	760.0	985.2	1533.4
C	701.2	873.6	1106.6	1663.7
D	600.9	760.8	999.8	1637.8
E	747.2	937.1	1219.6	1938.9
F	625.0	787.7	1032.3	1692.8
G	648.0	817.4	1042.0	1594.0
H	927.8	1117.1	1349.1	1901.5

TABLE 6.- FUEL CONSUMPTION (kg) REQUIRED TO ACCELERATE AND CLIMB

Velocity distribution	Final altitude, km			
	9.15	10.769	12.2	13.7
A	226.1	259.0	298.1	375.7
B	224.9	257.4	295.9	372.7
C	257.0	291.3	330.2	407.7
D	225.0	257.6	296.9	382.1
E	271.8	309.3	354.8	449.8
F	231.0	263.9	303.9	391.7
G	256.0	293.5	334.3	414.1
H	395.6	438.9	483.9	564.6

TABLE 7.- HORIZONTAL DISTANCE FLOWN (km)

Velocity distribution	Final altitude, km			
	9.15	10.769	12.2	13.7
A	100.5	130.2	176.6	295.3
B	96.9	125.6	170.5	287.4
C	90.5	118.2	162.9	280.2
D	97.1	125.4	170.2	295.8
E	91.0	119.1	166.8	302.4
F	92.6	120.6	165.6	294.1
G	124.8	160.7	209.4	331.8
H	202.0	244.3	296.2	419.6

TABLE 8.- COMPARISON OF UNSTEADY DISTRIBUTION AND STEADY DISTRIBUTION

FOR MINIMUM TIME⁴ - ACCELERATION AND CLIMB FROM $Z = 0$, $V = 274 \text{ km/h}$ TO $Z = 13.7 \text{ km}$, $V = 805 \text{ km/h}$ -

TOTAL AND PARTIAL TIME (sec)

Branch of path (fig. 13)	Velocity distribution	
	A	D
Horizontal flight accelerated to $Z = 0$	31.3	28.4
Horizontal flight accelerated to $Z = 10.769 \text{ km}$	25.1	-----
Horizontal flight accelerated to $Z = 13.7 \text{ km}$	22.1	339.5
Climb	1442.8	1269.9
Total time	1521.3	1637.8

⁴Limited to unsteady (steady) distribution minimum time with $V = V(Z)$ which solves the problem of the optimum with tangential accelerations included (disregarded).

TABLE 9.- ACCELERATION FACTORS F_a FOR SOME TYPICAL
VELOCITY DISTRIBUTION

Altitude, km	Rate of Climb		
	Minimum time, A	Minimum fuel consumption, B	Steepest, C
0	0.963	0.970	0.962
3.05954	.961	.945
6.10938	.944	.918
9.15909	.914	.875
10.769887	.886	.843
10.769915	.880	.815
12.2885	.837	.784
13.7844	.776	.745

TABLE 10.- RATE OF CLIMB - MINIMUM TIME - COMPARISON OF THE VALUES OF
 OPTIMUM VELOCITY AND RATE OF CLIMB COMPUTED BY ASSUMING
 $Q = \text{CONST.}(V, V_z)$ WITH THOSE DEFINED BY CONSIDERING
 THE CHANGE IN WEIGHT OF THE AIRPLANE DUE TO FUEL
 CONSUMPTION (V' , V'_z)

Z, km	V, km/h	V', km/h	V _z , m/s	V' _z , m/s
0	537.2	537.1	28.6	28.7
3.05	567.0	566.5	22.6	23.0
6.1	604.8	603.5	17.0	17.7
9.15	655.3	652.1	11.6	12.5
10.769	690.1	685.1	8.9	9.8
10.769	733.9	728.9	8.7	9.6
12.2	760.6	752.5	4.8	5.8
13.7	799.5	785.5	1.2	2.3

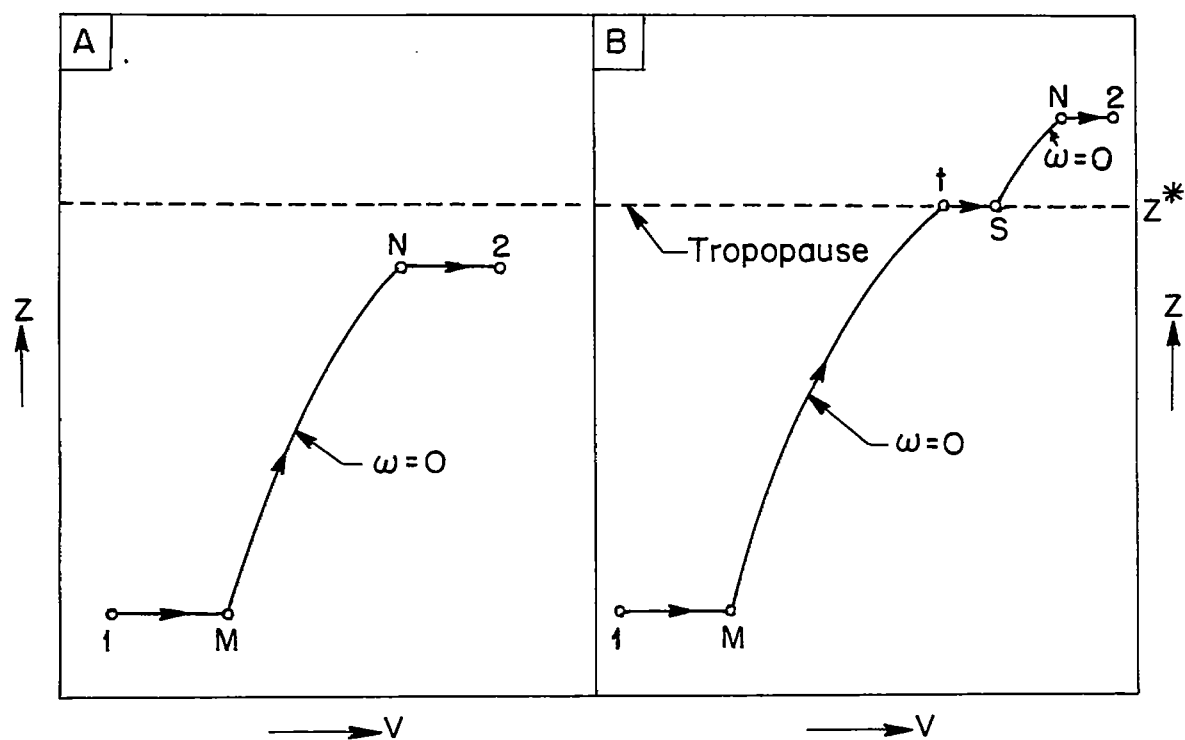


Figure 1.- Distribution of optimum velocity for minimum time climb;
case 1, $Z_1 \leq Z \leq Z_2$.

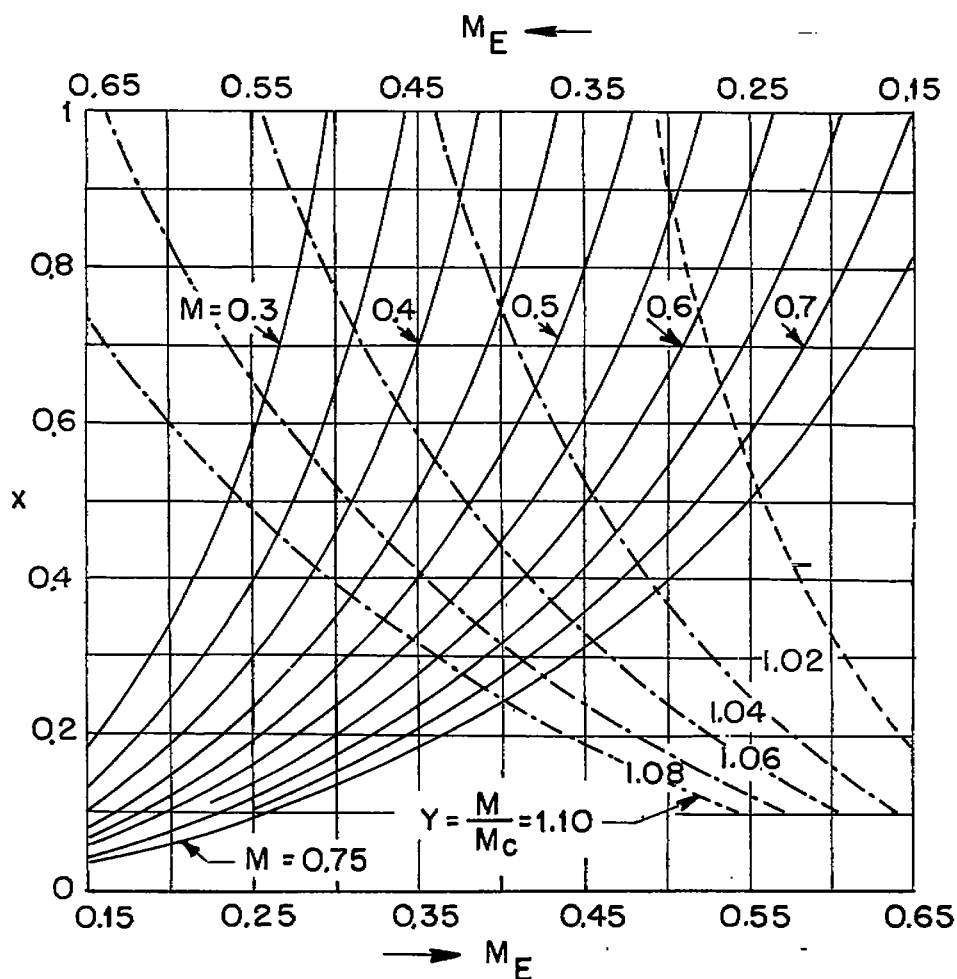


Figure 2.- Optimum Mach number for minimum time climb (troposphere).
 The lower scale of M_E is used to determine $M = f(x, M_E)$; the upper
 scale of M_E for $Y = M/M_c = f(x, M_E)$.

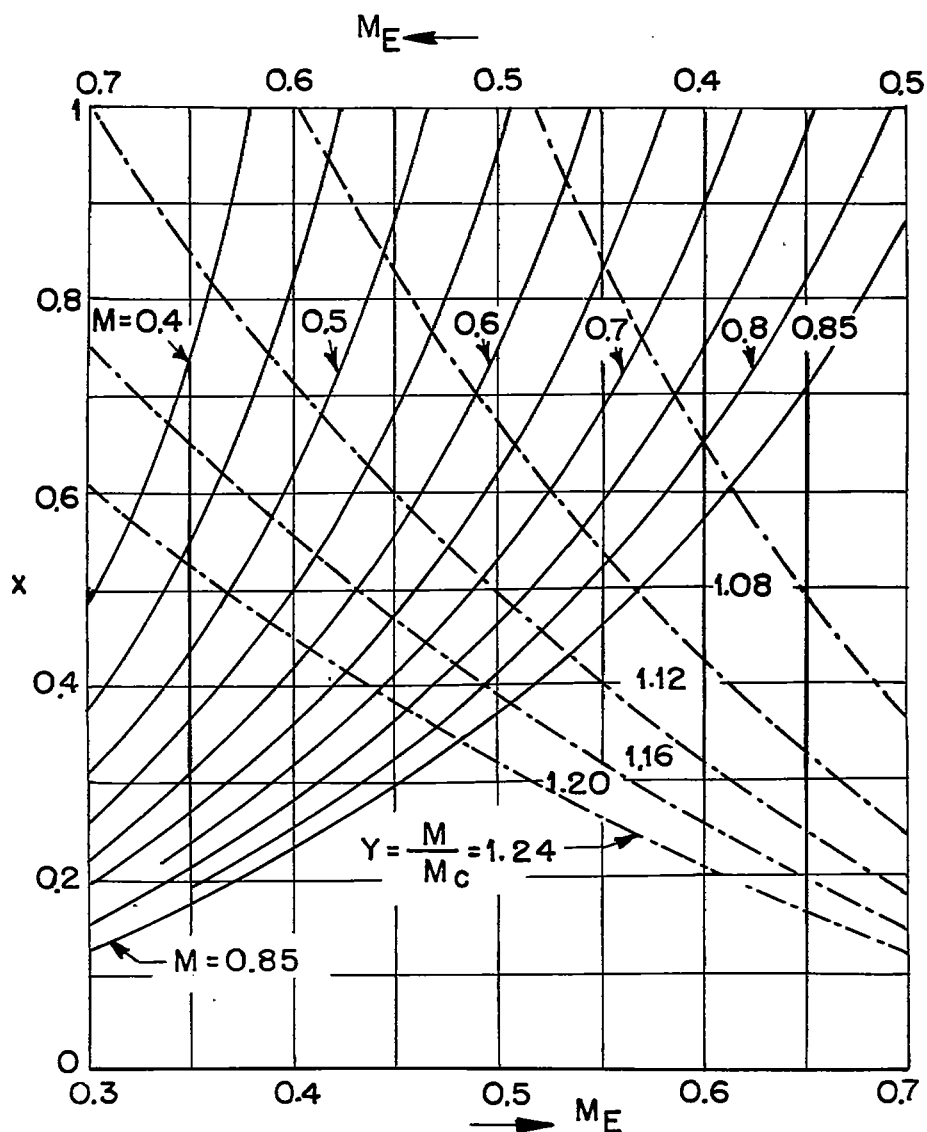


Figure 3.- Optimum Mach number for minimum time climb (stratosphere).
 The lower scale of M_E is used to define $M = f(x, M_E)$; the upper for
 $Y = M/M_C = f(x, M_E)$.

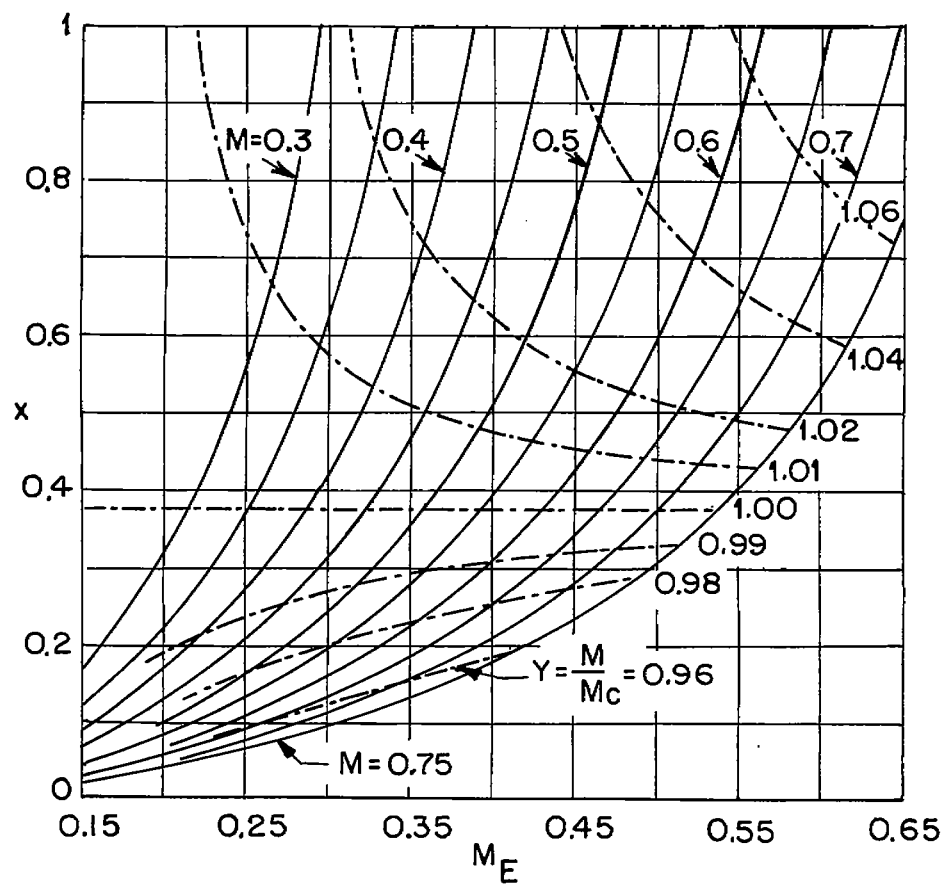


Figure 4.- Optimum Mach number for minimum consumption climb in the troposphere.

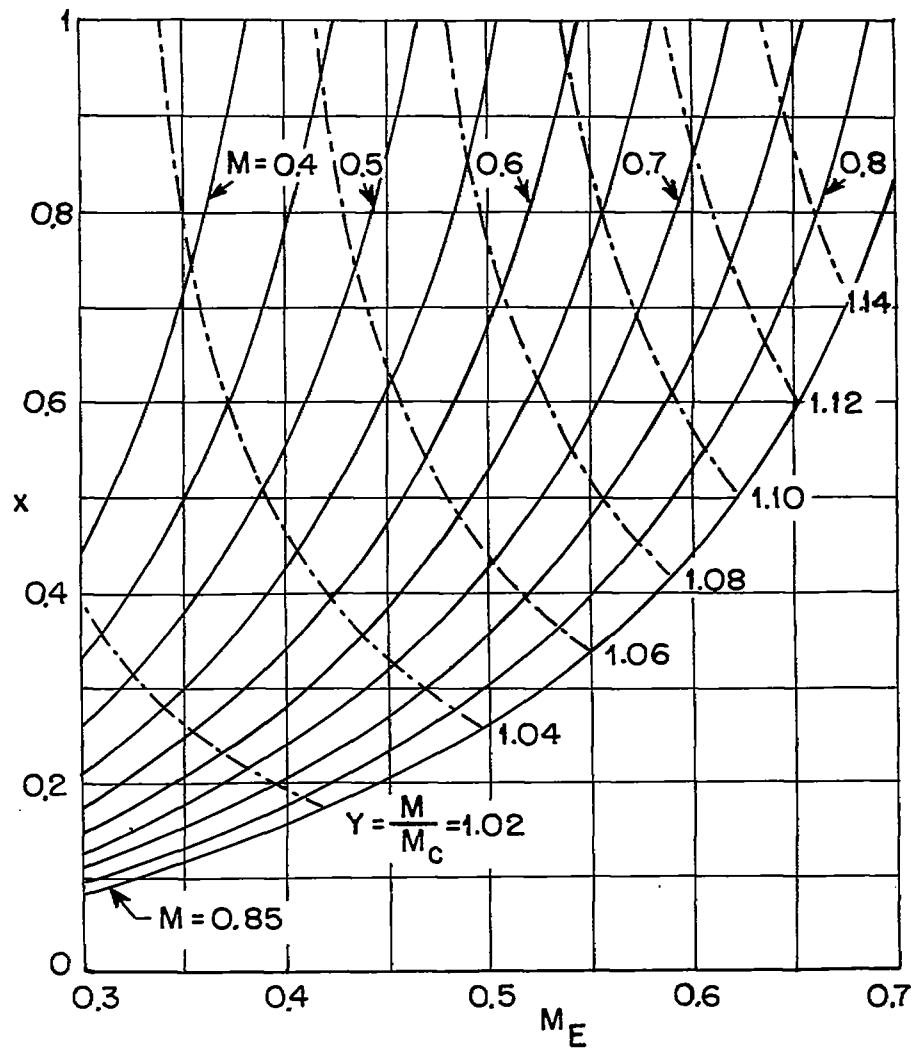


Figure 5.- For minimum consumption climb in the stratosphere.

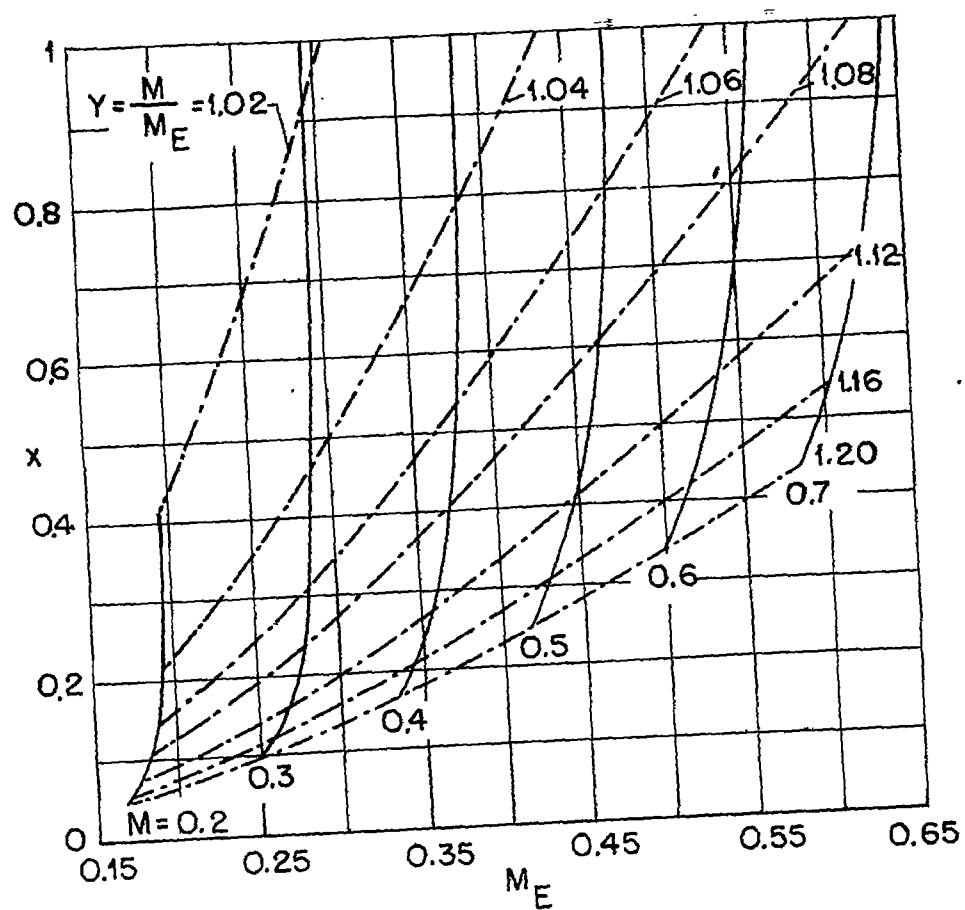


Figure 6.- Optimum Mach number for steepest climb (troposphere).

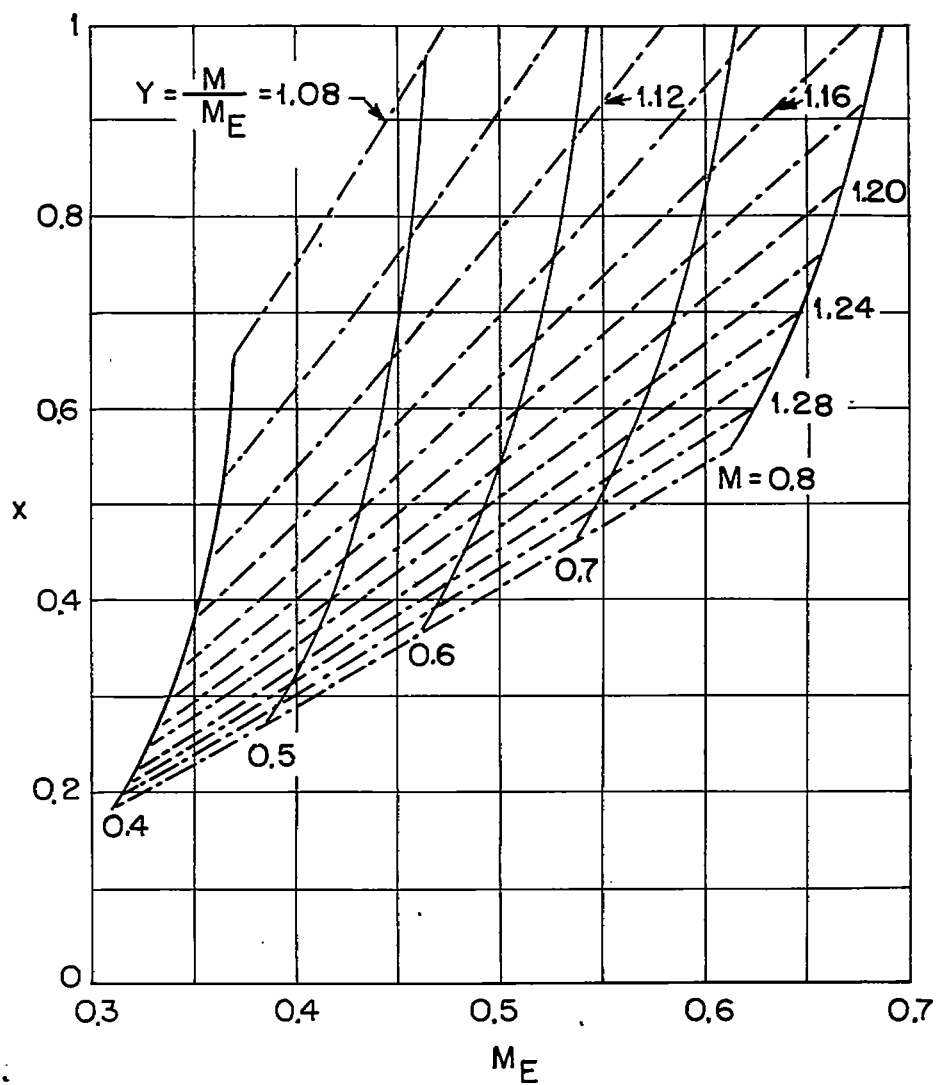


Figure 7.- Optimum Mach number for steepest climb (stratosphere).

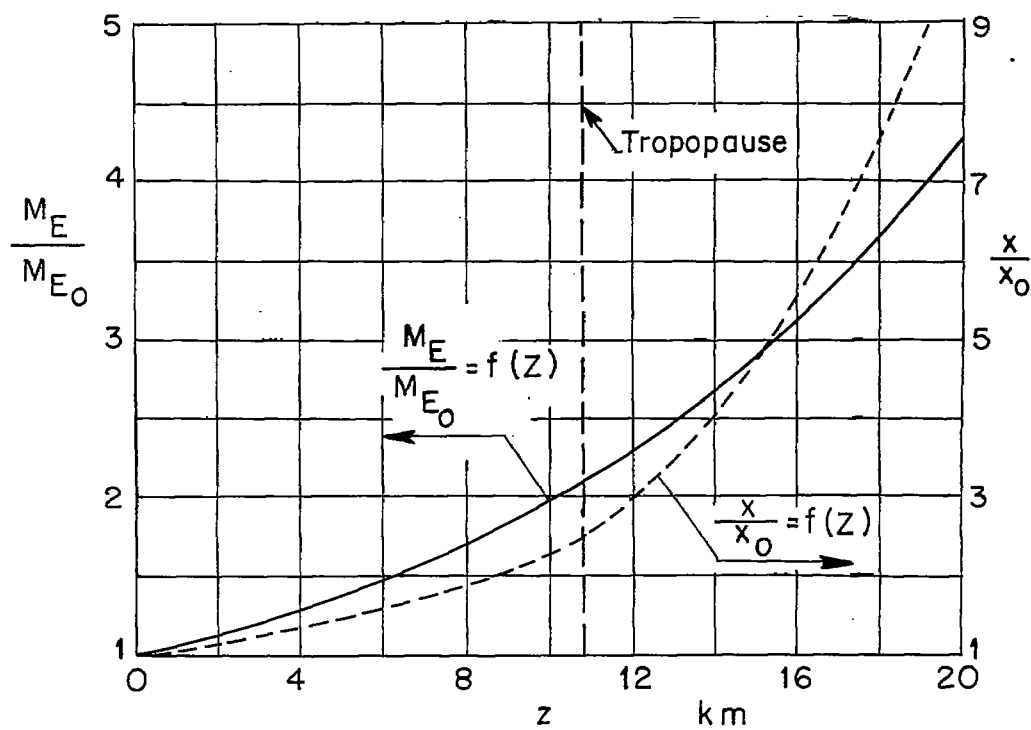


Figure 8.- Functions $M_E/M_{E0} = f(Z)$; $x/x_0 = f(Z)$.

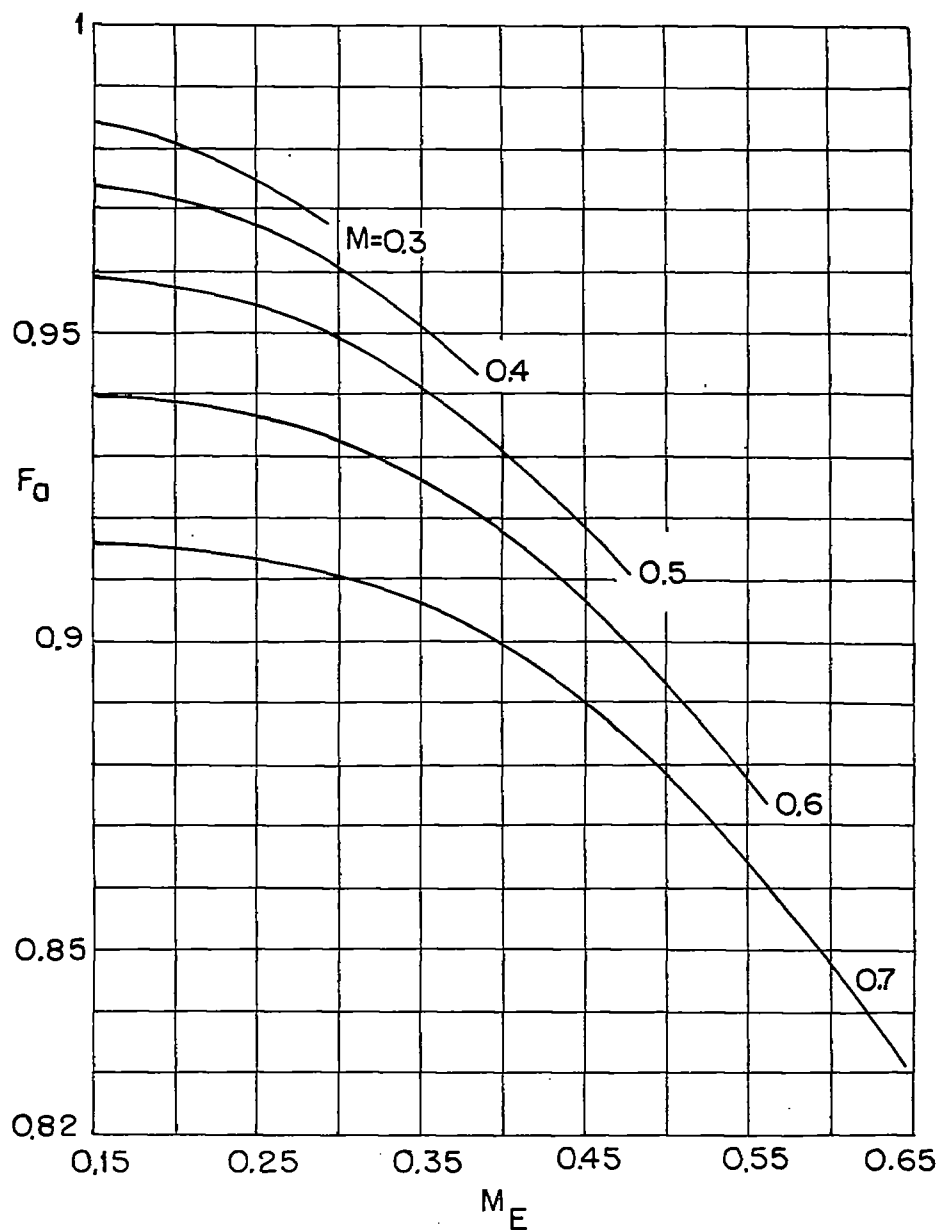


Figure 9.- Acceleration factor $F_a = f(M, M_E)$ for minimum time climb (troposphere). M = optimum Mach number for minimum time climb read from figure 2.

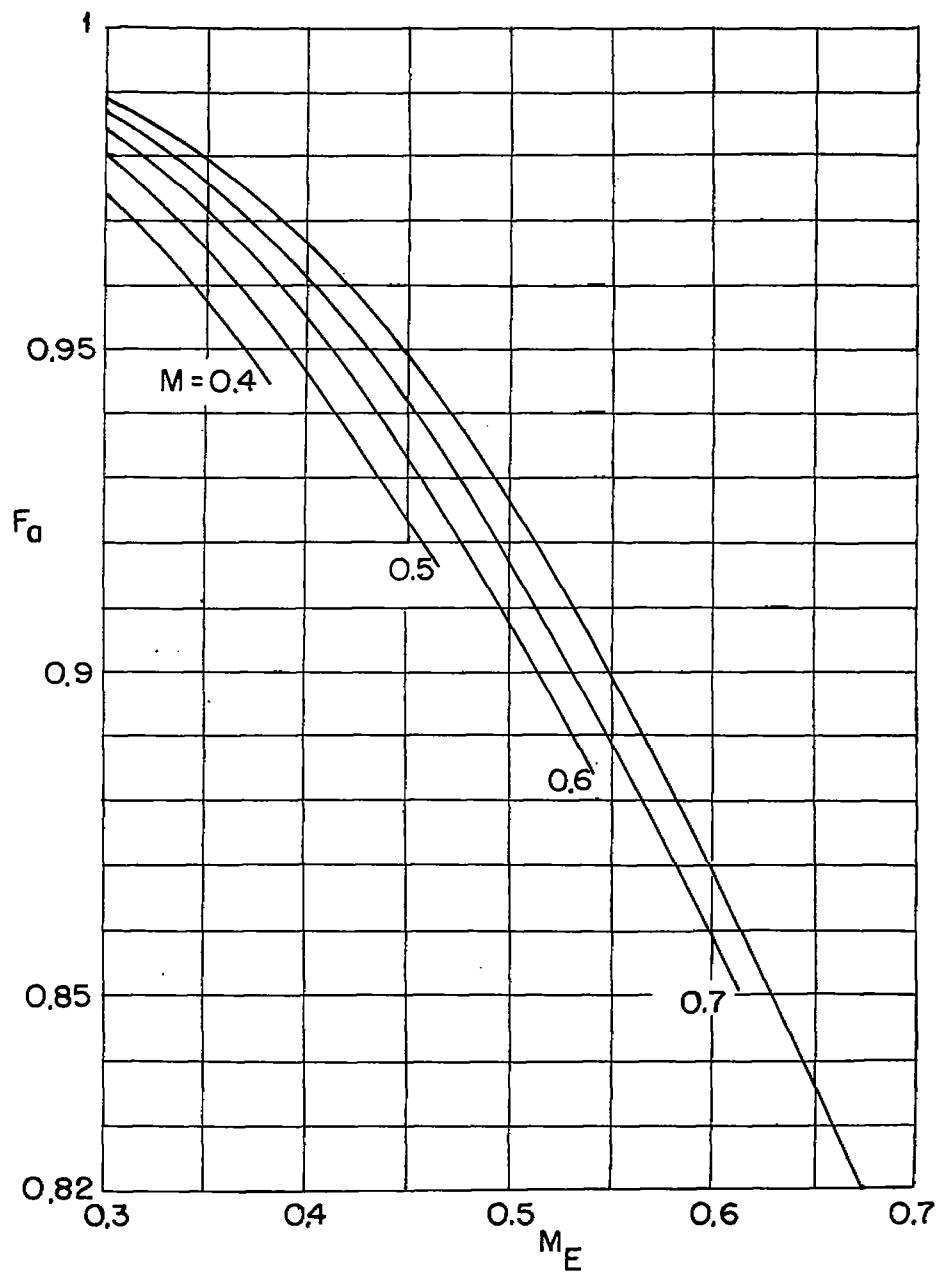


Figure 10.- Acceleration factor $F_a = f(M, M_E)$ for minimum time climb (stratosphere). M = optimum Mach number for minimum time climb read from figure 3.

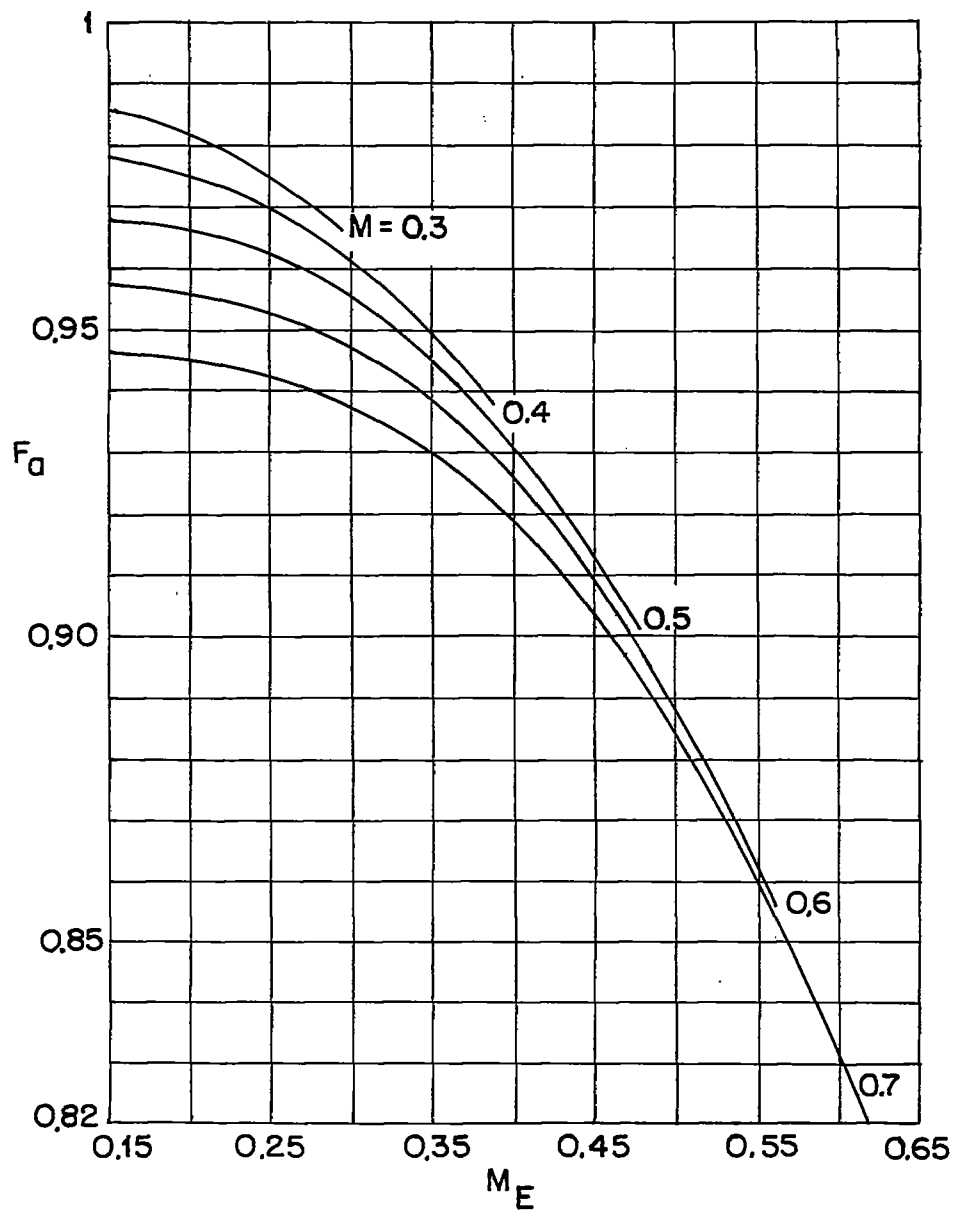


Figure 11.- Acceleration factor $F_a = f(M, M_E)$ for climb of minimum fuel consumption (troposphere). M = optimum Mach number for climb with minimum fuel consumption, obtainable from figure 4.

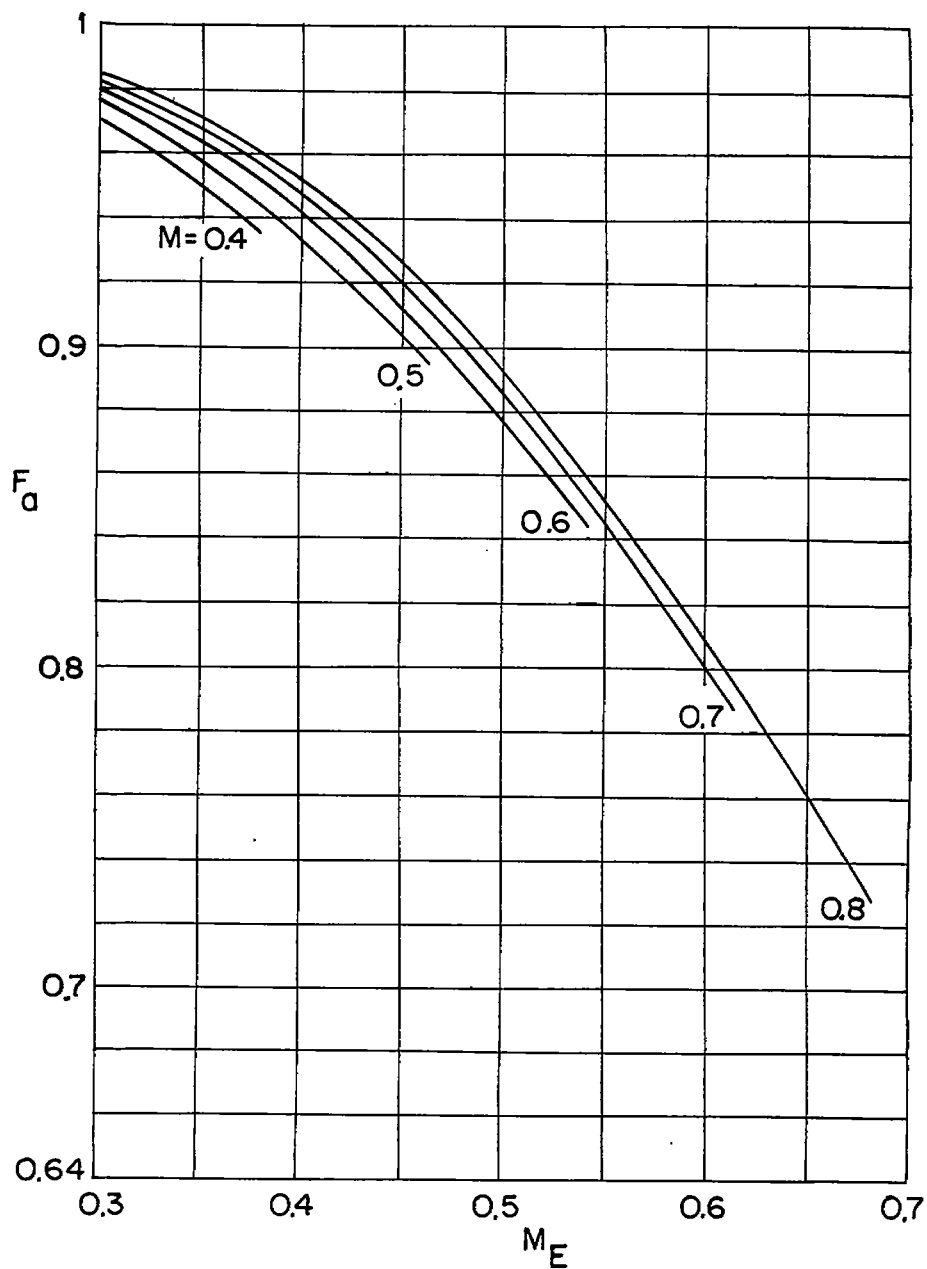


Figure 12.- Acceleration factor $F_a = f(M, M_E)$ for climb of minimum consumption (stratosphere). M = optimum Mach number for climb of minimum consumption read from figure 5.

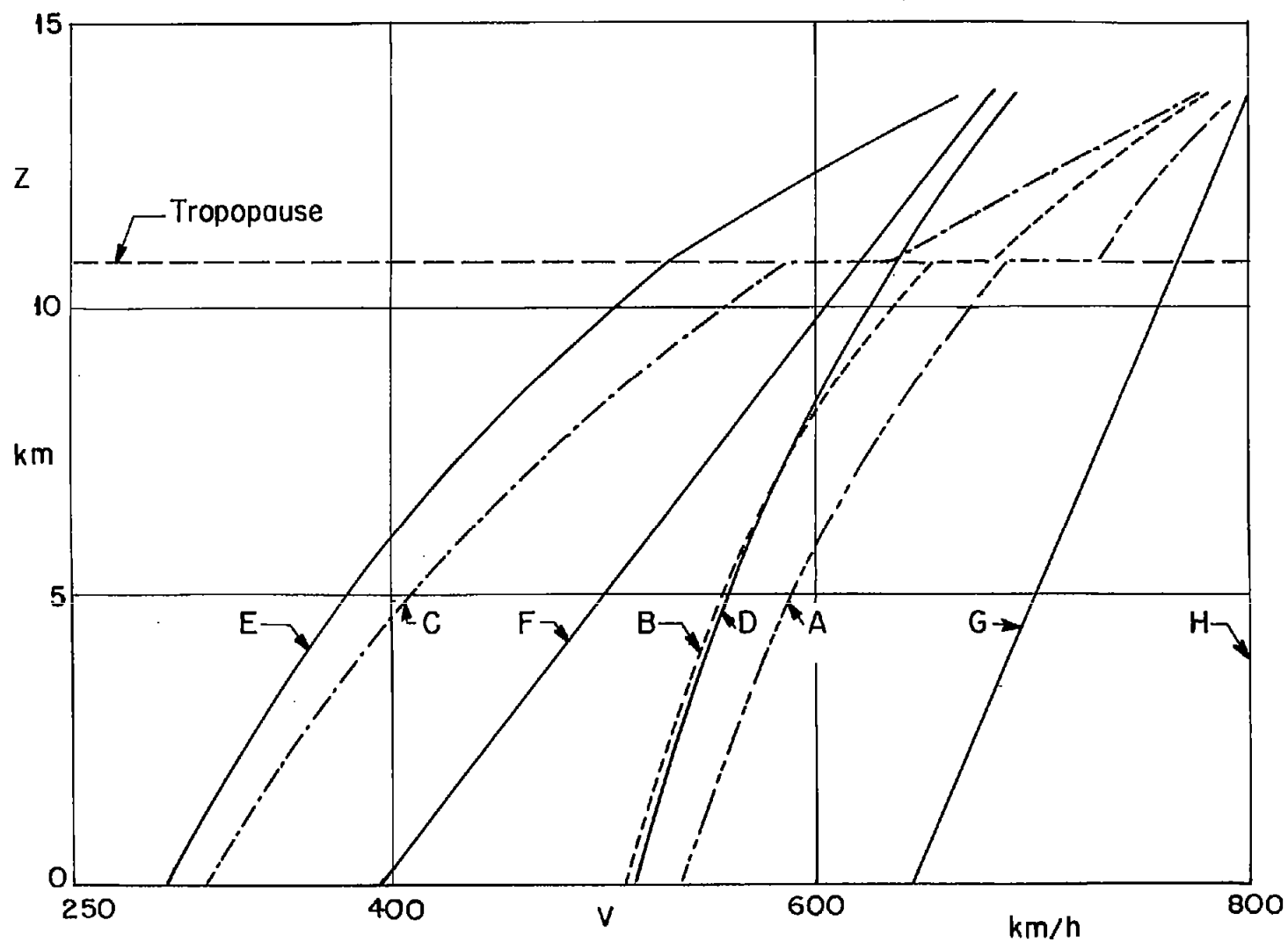


Figure 13.- Comparison of different velocity distributions.